Measurement principal.

Here we discuss how to determine "a" from the data of proton time of flight, T_p, vs electron kinetic energy, T_e. The electron time of flight is 1000 times smaller than T_p \sim 5 \mu\text{sec} and will be neglected for the purposes of discussion. The allowed phase space for proton momentum squared, z=p^2, T_e and is shown in Figure 1. For a = 0, the distribution of yield is uniform in p^2 at each value of T_e.

\[
p[T_{cut}]^2, (p^2T_e + pT_e)^2 \text{ and } (p^2T_e - pT_e)^2 \text{ vs. } T_e
\]

Figure 1. Available phase space for neutron beta decay vs. (x-axis) electron kinetic energy (T_e) and (y-axis) proton momentum squared (p^2). The black horizontal line shows the p^2 cut imposed by T_p < 25 \mu\text{sec}. Events below the line are not detected.

In figure 2, we show plots of yield vs. p^2 for representative T_e. If our spectrometer measured p^2 perfectly, "a" could be read off from the data. The (slope of the proton yield) / (p^2 range) is \beta "a". "a" could be determined from the yield plots without any additional knowledge of the magnetic field. As shown in Figure 1, at electron energy T_e = .236 MeV where the electron and neutrino momenta are equal, the yield extends down to p^2=0. the lower limits are >0 for other energies. The upper limit is an increasing function of T_e.

The relationship between T_p and p is given by
\[
T_p = \int_0^{s(L)} \frac{m}{c \sqrt{p^2(1 - (1 - u^2)B(s)/B[0]) + 2 me(\phi(s) - \phi[0])}} \, ds
\]

\(s = \) arc length along a magnetic field line
\(m = \) proton mass 
\(e = \) proton charge 
\(\phi = \) electric potential 
\(p = \) initial proton momentum 
\(B = \) Magnetic field strength 
\(u = \cos[\theta_{p-B}]\)

The motivation for the spectrometer design shown in Figure 2 is
1. The spectrometer is azimuthally symmetric about a magnetic field in the z direction.
2. Neutrons decay in the region of large magnetic field. The decay protons (and electrons) spiral around a magnetic field line (the guiding center approximation)
3. The momentum of the proton rapidly becomes parallel to the magnetic field. Between the values of z after the field expansion and before the electric field, the proton time of flight, \(T_p\), is \(\sim \frac{L m}{|p_p|}\), and this contribution dominates the total time of flight.
4. This relationship is modified by the terms involving \(u\). The time up to the electric field region is a product, \(T_p= \frac{L m}{|p_p|} f[u]\). In this approximation the yield

\[Y[y] = \frac{1}{x_{\text{min}}} \int_{x_{\text{min}}}^{x_{\text{max}}} X[x] Q[y/x] \, dx/x \]

\(x_{\text{max}}\) is modified by a function of \(u\) that is independent of \(p\). \(Q[r]\) is the probability density function of \(f[u]\).

5. The presence of the electric field changes this relationship (as shown by eq. 1) and causes \(F\) to become a function of both \(r\) and \(y=p^2\).
Figure 2. shows the yield vs. $\mathbf{p}_2$ for representative electron energies. The upper and lower edges correspond to the $\mathbf{p}$ phase space in Figure 1. The location of the edges is given by a single function of $T_e$ and the locations can be used to verify that the spectrometer fields have been measured correctly. The edges are sharp and the loss of information from determining the relationship between $T_p$ and $\mathbf{p}$ will be small.

![yield vs $\mathbf{p}^2$ for $T_e = .075$, .236, .450, and .700 MeV](image)

Figure 2. A plot of proton yield for 4 different electron kinetic energies for an ideal spectrometer. The spectrometer is assumed to have perfect time resolution. It is assumed that, TOF $\mathbf{p}_p = \text{constant}$ as would be the case if the electric field were zero. "$a"=-0.11$ is assumed. If $a$ were 0, all the distributions would have a slope of 0.

For a spectrometer that has both electric and magnetic fields, the resolution function will have a non-zero width, ~ 10% wide for a spectrometer optimized for $N_{ab}$, and the resolution function will depend on $y$ as well as $r$. First we show that it is possible to transform $T_p$ in such a way that the resolution function is nearly independent of $y$. A narrow resolution function gives a small increase in the uncertainty in "$a"$ above the uncertainty that would be obtained if the spectrometer fields were known perfectly. Then we investigate how the width of the resolution function depends on properties of the spectrometer such as the field expansion ratio, $r_B$, (the ratio of the decay field to the drift field), the curvature of the decay field, $\gamma = -\frac{d^2 B}{dz^2} / B$, and the width of the beam along $z$, $dz$. We find that, the width of the resolution decreases as $\gamma$ increases and decreases as $r_B$ increases. For reasonable values of $\gamma$ and $r_B$, a beam width of $dz=2$ cm, (which gives a count rate of 400 Hz) produces a small increase in the width over that from $\gamma$ and $r_B$. 
The strategy for making the relationship between $p$ and $T_p$ independent of $p$ is to calculate $T_p$ vs. $p$ for $u=1$, proton direction along a field line. We then solve this relation $T_p[p,1]$ for $p$ for each experimental value of $p=q$.

Figure 2.1 the product of proton time of flight and proton momentum vs. proton momentum. If the electric field were absent, the product of momentum and TOF would be constant.
We apply the transformation and calculate the probability density function (PDF) for a uniform distribution of $u$, $-1 < u < 1$, and show that to a good approximation the PDF's are independent of $p$. We show results for $p=.9$ and $.1$ of its maximum value.
Figure 3. Spectrometer resolution function PDF of deduced proton momentum for p =119 and 1,069 MeV/c. The widths differ by 2% of their values. The variation of the product of proton time of flight and proton before the mapping is ~ 15% The variation of the fractional width of the 1/Tp is similar. Application of the mapping reduces the width variation by a factor of 7.5 over the full range of proton momentum.

We calculated the width of the resolution function for field expansion ratios, rB, of 20, 30, 50, and 100. The results are shown in Figure 4. Over a range of reasonable values, the width \( \sim \frac{1}{rB^{0.4}} \). This behavior can be understood from eq. 1. The protons spend most of their time in the drift region, where \( B(s)/B(0) \sim 1/rB \). A large value of rB decreases the influence of the \((1-u^2)\) term that causes a variation of the velocity in the drift region. If the electric field were zero, the width variation would be \( \sim \frac{1}{\sqrt{rB}} \).

![log-log plot of \( \Gamma \) vs. rB](image)

Figure 4. Plot of the width of the response function vs. the field expansion ratio. The width decreases as the field expansion ratio increases.

We studied the dependence of the width of the resolution function on the rate of decrease of the magnetic field from its maximum value at z=0 to the drift region. This was done by changing the radius of the split pair of coils that create the field while keeping the ratio of the separation to the diameter, 0.3, constant. Figure 5. shows a plot of the width of the resolution function vs. coil diameter for a field expansion ratio of 100. The resolution improves as the coil diameter is decreased. This behavior can also be understood from eq. 1. The z-component of the proton velocity is \( \sim u \) before the field expansion. For large coil radii, the proton spends more time before the angle between the momentum is decreased by the field expansion. However, the coil diameter and separation can not be decreased.
indefinitely because with no separation no neutrons can enter the decay region. A radius of 5 cm and a separation of 3 cm allow a beam 2 cm along z and a count rate of 400 Hz.

Finally, we assumed a field expansion of 100 and a coil radius of 5 cm and a separation of 3 cm and calculated the change in the width of the spectrometer as a function of the full neutron beam width along z, Δz, for Δz= 0, 1, and 2 cm. The width of the response was constant to within 5% of its value. The PDF Q[r] of r, the factor that multiplies p^2 in eq. 1, is given in Figure 6. The width of Q[r] is 8.48 %. Q[r] is the distribution of p^2, and is approximately twice as wide as the distributions of p above. Since the width of Q[r] is approximately constant, we chose Δz=2 cm, the largest beam that will fit between the decay field coils, in order to maximize the count rate, 400 Hz.
We start with the yields at a range of electron energies, like those shown in Figure 2. We smooth these yields to calculate the yields for the model spectrometer. A $z = p^2$ spectrum for electron energy 0.469 MeV is shown in Figure 7. We perform fits to these spectra using the trial function

$$Y_2[z] = Y[a(z - z_0)] + \left[ Y[(1+\varepsilon)\alpha(z - z_0)] - Y[(1-\varepsilon)\alpha(z - z_0)] \right]/2\varepsilon$$

This trial function includes a $p^2$ offset ($z_0$), a $z$ calibration error ($\alpha$), and width uncertainty ($\varepsilon$). We calculate the measurement matrix, $M$, at each energy. We form the yield-weighted measurement matrix by summing over energies. We then calculate the uncertainty in $a$ by fitting all 4 parameters. The results are:

Uncertainty in a per root event assuming perfect knowledge of the spectrometer = 2.3
Uncertainty in a per root event from the fitting procedure = 2.6

The error correlation matrix is:

$$
\begin{pmatrix}
1 & .136 & .247 & .403 \\
.136 & 1 & .162 & .474 \\
.247 & .162 & 1 & .796 \\
.403 & .474 & .796 & 1
\end{pmatrix}
$$

The parameter most strongly correlated with $a$ is the width of the response function. The best approach to minimizing the uncertainty in $a$ is to design a spectrometer that has a narrow response function as described above. We illustrate this point by increasing the width of the response function by a factor of 3. The results are shown in Figure 7b.
Figure 7a. Calculated yield for the spectrometer response function shown in Figure 6a.
Conclusions:
For an attainable spectrometer configuration, the yield vs. proton momentum squared and electron kinetic energy can be used to check the spectrometer response calculated determined from field measurements. If we carry out the checks in the commissioning phase we can validate the measured fields. We can then use the fitting procedure to constrain the spectrometer response. The fitting procedure increases the uncertainty in a from 2.3 to 2.6, a small price to pay in order to obtain a robust measurement of a from an in situ check of the measured fields.